Contents

Preface v

1. The Genesis of Quadratic Reciprocity 1
1.1 P. Fermat 1
1.2 L. Euler 3
1.3 A.-M. Legendre 6
1.4 C.-F. Gauss 9

2. Quadratic Number Fields 43
2.1 Quadratic Fields 43
2.2 Genus Theory 47
2.3 Genus Characters 52
2.4 The Lucas-Lehmer Test 56
2.5 Hilbert Symbols and K_2 60

3. Cyclotomic Number Fields 79
3.1 Cyclotomic Fields 79
3.2 Primality Tests 85
3.3 Quadratic Gauss Sums 93
3.4 Cyclotomic Units 97

4. Power Residues and Gauss Sums 111
4.1 Residue Symbols in Number Fields 111
4.2 Gauss's Lemma 114
4.3 Discriminants 116
4.4 Kummer Extensions 119
4.5 Characters of Abelian Groups 121
4.6 Sums of Gauss, Jacobi and Eisenstein 126

5. Rational Reciprocity Laws 153
5.1 L. Dirichlet 154
5.2 A. Scholz 160
5.3 E. Lehmer 161
5.4 Rational Quartic Reciprocity 164
5.5 Residue Characters of Quadratic Units 168

6. Quartic Reciprocity 185
6.1 Splitting of Primes 186
6.2 Quartic Gauss and Jacobsthal Sums 190
6.3 The Quartic Reciprocity Law 194
6.4 Applications 197
6.5 Quartic Reciprocity in some Quartic Fields 198

7. Cubic Reciprocity 209
7.1 Splitting of Primes 209
7.2 The Cubic Reciprocity Law 212
7.3 Sextic Reciprocity 217
7.4 Cubic Reciprocity in some Quartic Fields 220

8. Eisenstein's Analytic Proofs 235
8.1 Quadratic Reciprocity 236
8.2 Abel's Construction of Elliptic Functions 239
8.3 Elliptic Functions 247
8.4 Quartic and Cubic Reciprocity 251
8.5 Quadratic Reciprocity in Quadratic Fields 256
8.6 Kronecker's Jugendtraum 260
8.7 The Determination of Gauss Sums 265

9. Octic Reciprocity 289
9.1 The Rational Octic Reciprocity Law 289
9.2 Eisenstein's Reciprocity Law 292
9.3 Elliptic Gauss Sums 299
9.4 The Octic Reciprocity Law 304
9.5 Scholz's Octic Reciprocity Law 307

10. Gauss's Last Entry 317
10.1 Connections with Quartic Reciprocity 317
10.2 Counting Points with Cyclotomic Numbers 321
10.3 Counting Points with Jacobi Sums 326
10.4 The Classical Zeta Functions 328
10.5 Counting Points with Zeta Functions 335

11. Eisenstein Reciprocity 357
11.1 Factorization of Gauss Sums 357
11.2 Eisenstein Reciprocity for ℓ-th Powers 361
11.3 The Stickelberger Congruence 366
11.4 Class Groups of Abelian Number Fields 371

A. Dramatis Personae 411

B. Chronology of Proofs 413

C. Some Open Problems 418

References 419

Author Index 472

Subject Index 484